Wavelet analysis of microscale strains

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Abstract

Recent improvements in experimental and computational techniques have led to a vast amount of data on the microstructure and deformation of polycrystals. These show that, in a number of phenomena, including phase transformation, localized bands of deformation percolate in a complex way across various grains. Often, this information is given as point-wise values arrayed in pixels, voxels and grids. The massive extent of data in this form renders identifying key features difficult and the cost of digital storage expensive. This work explores the efficiency of wavelets in storing, representing and analyzing such data on shape-memory polycrystals as a specific example. It is demonstrated how a compact wavelet representation captures the essential physics contained in experimental and simulated strains in superelastic media.

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1. Introduction

Recent developments in material characterization techniques have led to new abilities in determining the microstructure and strain fields of heterogeneous media [1–3], and in particular shape-memory alloys [4,5]. These are often represented as images composed of pixels or voxels. (For a review on the dominant electron microscopy tools and the state of the art X-ray techniques the reader is referred to Ref. [6].) Similarly, improvements in computational methods have enabled accurate full-field simulations for the mechanical fields developing in polycrystalline aggregates [7–9]. Again, such fields are described on grids.

These capabilities bring with them massive amounts of produced information. The digital storage space required can reach terabytes, depending on the grid, time resolution, dimensionality and fields of interest for a single experiment or simulation. To exploit this information, it needs to be stored, retrieved and analyzed efficiently; this has become a challenge [10].

Further, a combination of the heterogeneity, nonlinear behavior and fundamental physics (equilibrium and compatibility) causes the stress and strain in these materials to localize into bands (plates) that percolate in complex patterns through the material. Such patterns are observed in the context of elastic composites with a complex microstructure [12], plastic polycrystalline alloys [13] and shape-memory alloys [14]. An understanding of the properties of the materials requires us to readily identify these bands, to understand their interactions and to infer their consequences. However, as noted above, much of the data generated in experiments is arrayed point-wise, in pixels, voxels and grids. This makes it difficult to identify and analyze such bands.

These observations give rise to the question: what is a suitable way to represent and analyze strain fields with localized features?

In this work, we explore the efficiency of wavelets in representing and analyzing strain data. By construction, these...
functions are compactly supported at different length-scales, in both frequency and spatial domains, and are efficient in the representation of heterogeneous fields with localized features. The local multi-scale nature of wavelets is reminiscent of the way in which strain evolves: namely, from small-scale patches to bands. This similarity is the motivation for utilizing wavelet analysis in the representation of microscale strains.

The canvas upon which this concept is presented is that of polycrystals undergoing martensitic phase transformation. Such materials are capable of recovering strains beyond their apparent elastic limit. This phenomena is known as super- or pseudo-elasticity. Variants of martensite induced by stress, admitting non-zero transformation strains, rearrange to accommodate deformation without application of additional stress. Upon unloading, the variants transform back to austenite, and the strains caused by the rearrangement are recovered [15]. When considering a polycrystal, on top of a kinematic incompatibility of the phase mixtures within each grain, the different orientations of neighboring grains give rise to an additional inter-grain incompatibility. In turn, an intricate evolution of strain, transformation and stress fields emerges [16,17,14]. Transformation, in particular, initiates in local regions at grains which are well oriented with the loading. The intergranular compatibility constraint dictates non-uniform strains, as images to analyze vibrations and full-field strains in the macroscopic stress–strain relation, using a numerical model of a polycrystalline aggregate. We complete the paper by summarizing its main conclusions and observations in Section 5.

2. Wavelet representation

A brief introduction to wavelet representation is given here. For a more comprehensive review, the reader is referred to Mallat [18] and Walnut [19]. Broadly speaking, the main idea of representing a function is twofold. First, the space is partitioned into a nested sequence of scales at different resolutions. The function is then described in terms of its average across the coarse scale and its details across finer scales only in regions where the function is changing. Using this multi-resolution representation, one can focus attention on those specific regions where the interesting features occur and ignore the rest. This is in contrast with Fourier analysis, in which all regions of space are treated equally. This difference is illustrated in Fig. 1.

Rigorously, wavelet representation consists of a mother wavelet function \( \psi(x) \) of vanishing integral \( \int_{\mathbb{R}} \psi(x) dx = 0 \) and a scaling father function \( \phi(x) \) of a unit integral \( \int_{\mathbb{R}} |\phi(x)| dx = 1 \), both of which are locally supported. These functions are uniquely related, such that the wavelet function is a linear combination of translations of compressed scaling functions, i.e., \( \psi(x) = \sum_{n \in \mathbb{Z}} b_n \phi(2x - n) \), \( b_n \in \mathbb{R} \).

The translations and dilations of \( \psi(x) \) and \( \phi(x) \) are defined as

\[
\psi_{j,k}(x) \overset{\text{def}}{=} 2^{j/2} \psi(2^j x - k), \quad j,k \in \mathbb{Z},
\]

\[
\phi_{j,k}(x) \overset{\text{def}}{=} 2^{j/2} \phi(2^j x - k), \quad j,k \in \mathbb{Z}.
\]

A nested structure of approximation spaces for the square integrable functions is obtained using the function sets introduced in Eq. (2):

\[
V_j = \{ \phi_{j,k}(x), k \in \mathbb{Z} \}, \quad \{0\} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset L^2(\mathbb{R}).
\]

The orthogonal complement of \( V_j \) within \( V_{j+1} \) is spanned by \( \{\psi_{j,k}(x)\} \), i.e.,

\[
V_{j+1} = V_j \oplus W_j, \quad W_j = \{\psi_{j,k}(x), k \in \mathbb{Z}\}.
\]

![Fig. 1. Illustrative comparison between Fourier and wavelet transforms.](image-url)
By setting an approximation level \( j_0 \) at which the scaling functions are considered, the wavelet series of a square integrable function \( f(x) \) reads
\[
 f(x) = \sum_{k} a_k \phi_{j_0, k} + \sum_{j > j_0, k} d_{j,k} \psi_{j,k},
\]
where \( a_k = \int_{\mathbb{R}} f(x) \phi_{j_0,k} \, dx \) are the averaging coefficients and \( d_{j,k} = \int_{\mathbb{R}} f(x) \psi_{j,k} \, dx \) are the detail coefficients. Intuitively, \( a_k \) gives the averaging of \( f(x) \) at the scale \( j_0 \) along the domain of \( \phi_{j_0,k} \), providing a blurred version of \( f(x) \), while \( d_{j,k} \) adds the details at finer scales across the domain of \( \psi_{j,k} \).

The relation between wavelet and scaling functions of adjacent levels can be written as a linear combination, say
\[
 \phi_{j,k}(x) = \sum_{m} h_m \phi_{j+1, 2k+m}(x), \quad \psi_{j,k}(x) = \sum_{m} g_m \psi_{j+1, 2k+m}(x).
\]
The filters \( h \) and \( g \) also relate the coefficients \( \{a_{j,k}\} \) and \( \{d_{j,k}\} \) to \( \{a_{j+1,k}\} \) and \( \{d_{j+1,k}\} \), according to
\[
 a_{j,k} = \sum_{m} h_m a_{j+1,2k+m}, \quad d_{j,k} = \sum_{m} g_m d_{j+1,2k+m}.
\]
Eqs. (8) and (9) provide basis for the fast wavelet transform scheme of linear cost for calculating coefficients of coarser scales [20]. This procedure is illustrated in Fig. 2.

An extension to two (and analogously more) dimensions is carried out by defining the following generalizations of the scaling and wavelet functions
\[
 \phi_{j,k,l}(x,y) = \phi_{j,k}(x) \phi_{j,l}(y), \quad \psi_{j,k,l}(x,y) = \psi_{j,k}(x) \phi_{j,l}(y) = \phi_{j,k}(x) \psi_{j,l}(y), \quad \psi_{j,k,l}^{(2)}(x,y) = \phi_{j,k}(x) \psi_{j,l}(y), \quad \psi_{j,k,l}^{(3)}(x,y) = \psi_{j,k}(x) \psi_{j,l}(y).
\]
Accordingly, the expansion of a two-dimensional function \( f(x,y) \) takes the form
\[
 f(x,y) = \sum_{k,l} a_{k,l} \phi_{j_0, k,l} + \sum_{m \neq j_0, k,l} d_{j,k,l}^{(m)} \psi_{j,k,l}^{(m)},
\]
where \( a_{k,l} = \int_{\mathbb{R}^2} f(x,y) \phi_{j_0, k,l} \, dx \, dy \), and \( d_{j,k,l}^{(m)} = \int_{\mathbb{R}^2} f(x,y) \psi_{j,k,l}^{(m)} \, dx \, dy \).

Two main properties make wavelets advantageous in efficient representations of functions. First, few detail coefficients are needed to represent \( f \) in regions where it is smooth [19]. The actual number depends on the mother wavelet, and in particular its number of vanishing moments \( (\psi(x) \) has \( m \) vanishing moments, meaning that \( \int_{\mathbb{R}} x^{m-1} \psi(x) \, dx = 0 \)). Second, the \( L_2 \) norm of a function,
\[
 \| f(x) \|_{L_2}^2 = \int_{\mathbb{R}} f(x)^2 \, dx,
\]
coincides with the \( L_2 \) norm of its wavelet expansion coefficients vector, defined as
\[
 \| f_{j,k}(x) \|_{L_2}^2 = \sum_{k} a_k^2 + \sum_{j,k} d_{j,k}^2,
\]
(see Ref. [21], chapter 5 and reference therein). This implies that removing from the wavelet series functions with detail coefficients that are negligible in magnitude will introduce only a small error in an \( L_2 \) norm.

After choosing a wavelet family and a particular level \( j_0 \) of approximation, an efficient representation of \( f(x) \) is achieved by employing a thresholding process based on Eq. (16) as follows. Initially, the function is expanded in wavelet series. Subsequently, the expansion is truncated by keeping a fixed number of detail coefficients with the greatest magnitude. Note that all the averaging coefficients are kept during this process.

In the sequel, we will employ this procedure to functions that are described in terms of discrete values at pixels in a grid. We will measure the performance of the procedure in terms of a truncation ratio and a norm recovery. The former is the ratio between the number of coefficients retained in the wavelet expansion after thresholding and the number of pixels; the latter is defined as the ratio between the square of the \( L_2 \) norm of the wavelet coefficients vector after thresholding and the \( L_2 \) norm of the components of original field, i.e.,
\[
 \text{norm recovery} = \frac{\| \Phi \|_{L_2}^2}{\| \Phi \|_{L_2}^2},
\]
where \( \| \Phi \|_{L_2}^2 = \sum_{m \in \mathbb{N}} |f(n_0)|^2 \) and \( \{N\} \) are the pixels.

For completeness, we note that, in data compression, an additional coding step is employed [22] for storage purposes. This is beyond the scope of this paper, as our main goal is to examine the efficiency of wavelets in capturing the information in mechanical fields with localized features.

### 3. Representation of experimental data from superelastic media

The performance of the thresholding process described in the previous section depends greatly on the function of

\[
\{a_{n,k}\} \rightarrow \{a_{n-1,k}\} \rightarrow \{a_{n-2,k}\} \rightarrow \{a_{n-3,k}\} \rightarrow \cdots \rightarrow \{a_{0,k}\}
\]
\[
\{d_{n-1,k}\} \rightarrow \{d_{n-2,k}\} \rightarrow \{d_{n-3,k}\} \rightarrow \cdots \rightarrow \{d_{0,k}\}
\]

Fig. 2. A schematic illustration of the fast wavelet transform calculation. Given the set of averaging coefficients \( \{a_{n,k}\} \) at level \( n \), the coefficients at coarser levels are calculated recursively.
interest. Specifically, it depends on the way in which the information about the function is distributed among its wavelet coefficients. In this section we demonstrate how this process is useful in representing experimental data from strains of phase-transforming specimens.

To this end, we consider the full-field strain maps obtained by Daly et al. [17]. In their work, sheets of nitinol were subjected to uniaxial tension under displacement control. The local strains fields were extracted using the digital image correlation technique. In the sequel, we analyze the data samples on the 39×351 pixels at the center of the extracted images.

We apply the thresholding process to the axial (\( \varepsilon_a \)) and transverse (\( \varepsilon_{tr} \)) components of the strain. Throughout the calculations, we used five scales of detail coefficients. By way of an example, we retain three different amounts of wavelet coefficients, namely, 10, 5 and 3% of the number of pixels in the original images; that is, we set three truncation ratios and measure the performance in terms of the norm recovery.

Different families of wavelets were examined. Symlet2, an orthogonal family having two vanishing moments [23], demonstrated the highest scores of norm recovery. Its performance at representative overall axial strains \( \bar{\varepsilon}_a = 1.34, 1.93 \) and 2.94%, together with those of the Haar and Daubechies4 wavelet families, are given in Table 1. More than 99.94% of the \( L_2 \) norm is recovered, even when only 3% of the number of pixels are used. Similar performance scores were obtained when thresholding wavelet expansions of the transverse strain, but are not given here for brevity.

The original axial and transverse strain maps are displayed in the first and third columns of Fig. 3. Their truncated counterparts at a truncation ratio of 3% are shown in the second and fourth columns. The similarity between the fields, and in particular the information about the localized bands, is evident.

We conclude this section by demonstrating the pointwise ability of the truncated fields to recover the strain. To this end, we consider three different evolution states of transformation, namely, upon onset (\( \bar{\varepsilon}_a = 1\% \)), when halfway evolved (\( \bar{\varepsilon}_a = 2.6\% \)) and when macroscopically saturated (\( \bar{\varepsilon}_a = 5\% \)). The corresponding original and truncated strain fields at a truncation ratio of 3% are depicted in Fig. 4(a). We show the strain along the centerline of these fields in Fig. 4(b). The continuous and dashed curves correspond to the strain extracted from the original and truncated maps, respectively, and are in good agreement. As expected, the curves associated with the truncated fields are less smooth.

Finally, we note that similar analysis to additional sets of experimental data on strains of phase-transforming media reproduced the same performance, but are not given here to avoid redundancy.

4. Analysis of a computational model of a superelastic polycrystal

Aside from the data storage perspective, the question to be asked is if a truncated wavelet representation can qualitatively capture the interplay between the total strain, the transformation strain and the stress fields. We test this notion using the following model problem.

<table>
<thead>
<tr>
<th>( \bar{\varepsilon}_a (%) )</th>
<th>#terms</th>
<th>#pixels (%)</th>
<th>Haar (%)</th>
<th>Daubechies4 (%)</th>
<th>Symlet2 (%)</th>
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<td>5</td>
<td>99.961</td>
<td>99.989</td>
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<tr>
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<td>99.999</td>
<td>99.999</td>
<td></td>
</tr>
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<td></td>
<td>5</td>
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<td>99.996</td>
<td>99.999</td>
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</tr>
<tr>
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<td>3</td>
<td>99.983</td>
<td>99.968</td>
<td>99.994</td>
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</tr>
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</table>
4.1. A model problem of Richards et al. [14]

Consider a superelastic polycrystal consisting of identical grains at different orientations. The polycrystal is subjected to anti-plane shear. The displacement vector field $u$ has only one component, $u_z$, along the $z$-direction. The strain field $\varepsilon$ is its gradient

$$\varepsilon = \nabla u,$$  \hspace{1cm} (18)

having two non-vanishing components, $\varepsilon_x = \partial u_z / \partial x$ and $\varepsilon_y = \partial u_z / \partial y$. The strain is assumed to be a sum of an elastic strain, $\varepsilon'$, and a transformation strain, $\varepsilon''$, i.e.,

$$\varepsilon = \varepsilon' + \varepsilon''.$$  \hspace{1cm} (19)

The transformation strain is considered to be in the form

$$\varepsilon'' = \sum_{i=1}^{4} \lambda_i \varepsilon_i^t,$$  \hspace{1cm} (20)

where $\lambda_i$ is the volume fraction of the $i$th variant of the martensite stress-free strain $\varepsilon_i^t$. The stress $\sigma$ is given by a linear elastic relation

$$\sigma = C : \varepsilon',$$  \hspace{1cm} (21)

where $C$ is the elasticity tensor, and must satisfy the equilibrium equations

$$\nabla \cdot \sigma = \mathbf{0}.$$  \hspace{1cm} (22)

Substitution of Eq. (19) into Eq. (21) and, in turn, into Eq. (22) yields

$$\nabla \cdot (C : \nabla u) = \nabla \cdot (C : \varepsilon').$$  \hspace{1cm} (23)

Eq. (23) provides a different interpretation of the transformation strain, namely, the divergence of its product with the elasticity tensor as a source function in a Poisson-like equation for the displacement field.

The volume fractions $\{\lambda_i\}$ evolve according to the kinetic relation

$$\dot{\lambda}_i = \begin{cases} h'(d_i^k - d_i) & d_i^k \geq d_c, \\ h'(d_i^k + d_i) & d_i^k \leq d_c, \\ 0 & \text{else}, \end{cases}$$  \hspace{1cm} (24)

Here, $d_i^k = \sigma : \varepsilon'' - w$ is the driving force, $w$ being the chemical free energy. $1/h'$ is the transformation hardening coefficient and $d_c$ is the critical driving force for transformation. For brevity, the specific choice of material parameters and transformation strain variants are omitted; they can be found in Richards et al. [14].

The specimen is loaded under displacement control to 4% macroscopic strain in the $x$-direction, then unloaded back to zero strain. The evolution of the stress, total strain and transformation strain fields are calculated using a fast Fourier transform (FFT)-based algorithm. The coupled set of Eqs. (22) and (24) are solved in the following staggered scheme. Given a microstructure in terms of the volume fractions $\{\lambda_i\}$ at a current average strain $\bar{\varepsilon}_a$, a discretized version of Eq. (22) is solved when subjected to an additional increment in the overall strain – a loading step – based on the method of Michel et al. [24]. Subsequently, a discretized version of Eq. (24) is employed to update the volume fractions. This procedure is repeated until the whole loading is applied, which is then relieved.

To illuminate how transformation evolves, the local transformation strains at average axial strains $\bar{\varepsilon}_a = 0.5, 1.5$ and 4% along the loading path are depicted in Fig. 5, labeled by A, B and C, respectively. It is observed how transformation initiates in local regions (A) and progresses in non-uniform bands (B), perpendicular to the loading direction, until saturation (C). We note that a reversed trend is exhibited during unloading.

4.2. Representation of simulated strains

Similar to the process employed in the previous section, we first examine the ability of the wavelet expansion to
efficiently represent the developing transformation strain fields.

Specifically, we employ the thresholding process to the axial and transverse components of the transformation strain, namely, $e_{tx}$ and $e_{ty}$, respectively. The total number of wavelets retained was set to be 10% of the number of 512x512 pixels used in the original fields, i.e., the truncation ratio was set to 10%. Throughout the calculations, we used five scales of detail coefficients.

Exemplary scores of norm recovery at strains associated with points A, B and C were calculated. Scores at two additional points, D and E, associated with unloading states at overall axial strains $\varepsilon_x = 1.5$ and 0.5%, respectively, were also calculated. Again, performances of different families of wavelets were examined. The Symlet5 family demonstrated the highest results. Table 2 presents its scores, compared with scores of the Symlet2 family. A truncation of the Symlet5 expansion at a truncation ratio 1% was also carried out; its scores are also given in Table 2.

Remarkably, more than 98% of the $\ell_2$ norm is recovered when using the Symlet5 expansion with 10% of the number of pixels. When using the Symlet5 expansion with only 1% of the number of pixels, scores are deteriorated by 1–6% when thresholding $e_{tx}$, and by 8–12% when $e_{ty}$ is truncated. The reduction is more significant when thresholding the latter due to its rapid fluctuations, which require more detail coefficients for accurate representation.

For illustration purposes, the microscopic transformation fields evaluated using the 10% truncation of the Symlet5 wavelet expansion are shown in the right columns of each panel in Fig. 5.

From an image compression perspective, the wavelet transform serves as the basis for the JPEG 2000 standard, succeeding the JPEG standard, which is based on the discrete cosine transform (DCT). In the latter, as the name suggests, the basis functions used are cosines [25], making DCT intimately related to the Fourier transform. The compression process using DCT involves a division of the field into sub-blocks, applying the transform to each block, and keeping a fixed percentage of the greatest coefficients within each block. The reconstruction of the field is carried out by employing the inverse transform to each block and tailoring them all back together again.

To enable a comparison between the performance of these techniques, we follow the foregoing scheme to obtain a truncated DCT series, keeping the same number of coefficients in the DCT representation as in the truncated wavelet series. The results are given in the fifth and ninth columns of Table 2. It is shown how wavelet representation outperforms DCT by about 1–7%. The difference is greater when $e_{ty}$ is truncated. This is due to the highly discontinuous nature of the field, which reduces the rate of convergence of the DCT series in a manner reminiscent of the behavior of Fourier series.

Table 2
Norm recovery scores for $e_{tx}$ and $e_{ty}$ at exemplary loading states, using Symlet2 and Symlet5 wavelet expansions and DCT series, at a truncation ratio of 10%. Scores of the Symlet5 at a truncation ratio of 1% are also given.

<table>
<thead>
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<th>$\varepsilon_x$ (%)</th>
<th>$\ell_2$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(A) 0.5</td>
<td>99.76</td>
<td>99.72</td>
</tr>
<tr>
<td>(B) 1.5</td>
<td>99.73</td>
<td>99.63</td>
</tr>
<tr>
<td>(C) 4</td>
<td>99.80</td>
<td>99.73</td>
</tr>
<tr>
<td>(D) 1.5</td>
<td>99.48</td>
<td>99.25</td>
</tr>
<tr>
<td>(E) 0.5</td>
<td>99.45</td>
<td>99.22</td>
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From an image compression perspective, the wavelet transform serves as the basis for the JPEG 2000 standard, succeeding the JPEG standard, which is based on the discrete cosine transform (DCT). In the latter, as the name suggests, the basis functions used are cosines [25], making DCT intimately related to the Fourier transform. The compression process using DCT involves a division of the field into sub-blocks, applying the transform to each block, and keeping a fixed percentage of the greatest coefficients within each block. The reconstruction of the field is carried out by employing the inverse transform to each block and tailoring them all back together again.

To enable a comparison between the performance of these techniques, we follow the foregoing scheme to obtain a truncated DCT series, keeping the same number of coefficients in the DCT representation as in the truncated wavelet series. The results are given in the fifth and ninth columns of Table 2. It is shown how wavelet representation outperforms DCT by about 1–7%. The difference is greater when $e_{ty}$ is truncated. This is due to the highly discontinuous nature of the field, which reduces the rate of convergence of the DCT series in a manner reminiscent of the behavior of Fourier series.

Table 2
Norm recovery scores for $e_{tx}$ and $e_{ty}$ at exemplary loading states, using Symlet2 and Symlet5 wavelet expansions and DCT series, at a truncation ratio of 10%. Scores of the Symlet5 at a truncation ratio of 1% are also given.

<table>
<thead>
<tr>
<th>$\varepsilon_x$ (%)</th>
<th>$\ell_2$</th>
<th>$\ell_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) 0.5</td>
<td>99.76</td>
<td>99.72</td>
</tr>
<tr>
<td>(B) 1.5</td>
<td>99.73</td>
<td>99.63</td>
</tr>
<tr>
<td>(C) 4</td>
<td>99.80</td>
<td>99.73</td>
</tr>
<tr>
<td>(D) 1.5</td>
<td>99.48</td>
<td>99.25</td>
</tr>
<tr>
<td>(E) 0.5</td>
<td>99.45</td>
<td>99.22</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varepsilon_y$ (%)</th>
<th>$\ell_2$</th>
<th>$\ell_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) 0.5</td>
<td>99.76</td>
<td>99.72</td>
</tr>
<tr>
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4.3. Implications on physics

The foregoing results imply that wavelets are efficient for data storage purposes. An even more important question is whether such compact wavelet representation is sufficient for characterizing the interplay between the transformation strain, the total strain and the stress. In other words, we ask how the physics of the problem is affected by considering a compact wavelet approximation for the transformation strain.

To investigate this question, we examine how the evolution of strain and stress changes when replacing the transformation strain with its truncated wavelet series in the governing equations. Hence, we modify Eq. (23) by replacing \( \varepsilon' \) with its wavelet truncation \( \varepsilon_{th} \), namely,

\[
\nabla \cdot (C : \nabla u) = \nabla \cdot (C : \varepsilon_{th}).
\]

The truncation ratio for the truncated wavelet series was set to 10%. Specifically, at each loading step, the local total strain and stress fields are obtained by solving the approximated Eq. (25), and the volume fractions are updated accordingly. The thresholding process is applied to the updated transformation strain. The new wavelet truncation is used at the next loading step and the foregoing scheme is repeated until the whole loading has been applied and then removed. Thus, during this scheme, a different wavelet basis is used at each loading step. We refer to wavelets that constitute this basis as active wavelets.

The resultant macroscopic stress–strain relation obtained by solving Eq. (25) is given by the dashed curve in Fig. 6. This approximated relation is compared with the exact relation obtained by solving Eq. (23) and given by the continuous curve calculated in Richards et al. [14]. To numerically quantify the difference, we define the error of the approximated scheme as

\[
\text{err} = 100\% \cdot \frac{|\tilde{\sigma}_{x,th} - \tilde{\sigma}_x|}{|\tilde{\sigma}_x|},
\]

where \( \tilde{\sigma}_{x,th} \) and \( \tilde{\sigma}_x \) denote the axial macroscopic stress calculated using the thresholded and exact transformation strains, respectively. The average error is 3.6%, with a maximal value of 13.2%.

Qualitatively, the curves are shown to be in good agreement, and demonstrate the same essential trends: linear behavior at first, followed by a plateau originating about \( \bar{\varepsilon}_x = 0.5\% \) when transformation progresses and hardening beyond \( \bar{\varepsilon}_x = 2\% \), and vice versa upon unloading. When the fluctuations in the transformation field are less drastic at the elastic regime and close to macroscopic saturation, the error introduced with truncation is smaller. Hence, the difference between the corresponding sections of the curves is almost indistinguishable. Conversely, during the banded evolution of transformation there are severe fluctuations in the field. Consequently, the error introduced with truncation is more significant, and the difference between the curves attains its maximum.

A comparison between the set of remaining coefficients at successive loading steps reveals how the majority of them coincide. To quantify this similarity, we calculate the percentage of wavelets appearing in successive sets out of the whole set. On average, 96% of the wavelets that are active at a certain loading step are retained in the next one.

Furthermore, it appears that there are wavelets that are never active, in the sense that they are always removed during the thresholding process. To illustrate this observation, at each loading step we denote with black marks the support of retained wavelets in wavelet space. An activation map is composed of the union of all the maps at each loading step. By way of an example, Fig. 7 displays activation maps rendered by examining the support in the wavelet domain of wavelets associated with detail coefficients at the finest scale, which in our case corresponds to \( j = 5 \).

Specifically, Fig. 7(a)–(c) correspond to \( \{d_{5,k}^{(1)}\} \), \( \{d_{5,k}^{(2)}\} \) and \( \{d_{5,k}^{(3)}\} \), respectively. Recall that these coefficients are associated with different two-dimensional wavelet functions, defined in Eqs. (11)–(13), respectively. In a sense, the resultant set of active wavelets describes the principal “modes” in wavelet space. Conversely, white regions are associated with wavelets that are never active, or “modes” that can be discarded. Fig. 7(d) is the heat version of a section of Fig. 7(c), displaying the number of loading steps at which each wavelet was active. In other words, it shows how often each wavelet was used. This figure demonstrates that, within the set of active wavelets, some are used more often than others. Furthermore, it reveals that certain wavelets are retained throughout the whole scheme. We interpret those frequent wavelets as dominant modes.

We wish to explore the connection between these modes and the evolution of stress. To this end, Fig. 8 shows the macroscopic stress–strain relation of the spatial regions.
of active (dashed curves) and non-active wavelets (dash-dot curves); that is, the averages of the stress and strain are calculated separately over the physical domain of active and non-active wavelets. As a reference, the stress–strain relation of the whole domain is given as the continuous curve. It is observed how the overall stress at active modes is lower than at the domain of non-active ones. This result is to be expected as, in a way, regions associated with non-active wavelets undergo less transformation. Also, the overall strain in non-active modes is lower than in the domain of active ones, for the same reason.

4.4. Implementation of an adaptive wavelet scheme

Adaptive wavelet-based schemes are conceptually based upon refining an approximate solution by adding suitably chosen wavelets to the current approximation space. This is different, for instance, from finite element methods, in which refinement is obtained by adapting a mesh. The latter is carried out according to certain local error estimators. In adaptive wavelet methods, a refinement is carried out in the neighborhood of wavelets with coefficients of large magnitude, which play the role of error indicators. For a more thorough discussion, the reader is referred to Cohen et al. [26]. A preliminary realization of a scheme based on these fundamentals [27] to our setting was conducted. The computation time of our implementation was found to be less efficient than the FFT-based alternative. This is due to the known overhead in the computational cost involved with adaptive wavelet methods [28]. The overhead stems from the need to store not only the wavelet coefficients of the approximated solution and algebraic equation system, but also their scale and translation indices \( \{j, k, l\} \). This data has to be further manipulated and dynamically updated, resulting in a time-consuming computation.

5. Concluding remarks

The potential of wavelets in representing, storing and analyzing the transformation in superelastic polycrystals was examined. Experimental data sets of strain fields evolving in nitinol sheets when subjected to uniaxial tension were expended in wavelet series and truncated. It was shown how, by using 3% of the number of pixels in the original data, more than 99.94% of the \( l^2 \) norm of the field is retained. The reliability of the truncated fields in recovering the local behavior was also demonstrated. These observations suggest that wavelets can serve as a basis for an efficient method for storing and retrieving experimental data, rather than by pixels.

A numerical model of a superelastic polycrystal was used to demonstrate how a compact wavelet representation captures the interplay between the transformation strain, the total strain and the stress, in a case study of an anti-plane shear. The choice of the \( l^2 \) norm as a measure of how accurate the truncated field approximates the transformation strain was somewhat arbitrary, and might not be an optimal one from a physical perspective. Nevertheless, the wavelet coefficients retained according to this criteria contain the information needed for the evolution of the stress and strain fields. In particular, it was found how, by solving approximate equilibrium equations, given in terms of a truncated wavelet expansion of the transformation field, the essential characteristics of the macroscopic stress–strain relation are captured. Examination of the set of retained wavelets revealed a similarity between sets of active wavelets in adjacent loading steps. Furthermore, it was observed how some of the wavelets are always truncated, while others appear frequently. This distinction identifies those dominant as principal modes of the system in wavelet space. These observations suggest, for instance, that the principal wavelet modes can be used to obtain a coarse-grained
description of the system by considering a model using a reduced set of wavelets.

We recall that similar localized patterns are observed in other phenomena including polycrystalline plasticity and composites. Thus, while we have focused on shape-memory polycrystals, we believe that the lessons learnt in this work are applicable to a broad class of materials.

Finally, we examined the efficiency of adaptive wavelet methods in solving the governing equations. In our preliminary investigation, we found that the overhead in the computational cost makes it, at least at the current state of these methods, less efficient in terms of computational time.

Acknowledgments

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References