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ABSTRACT

The manipulation of mechanical waves is a long-standing challenge for scientists and engineers, as numerous devices require their control. The current forefront of research in the control of classical waves has emerged from a seemingly unrelated field, namely, non-Hermitian quantum mechanics. By drawing analogies between this theory and those of classical systems, researchers have discovered phenomena that defy conventional intuition and have exploited them to control light, sound, and elastic waves. Here, we provide a brief perspective on recent developments, challenges, and intricacies that distinguish non-Hermitian elastodynamics from optics and acoustics. We close this perspective with an outlook on potential directions such as topological phases in non-Hermitian elastodynamics and broken Hermitian symmetry in materials with electromomentum couplings.

I. INTRODUCTION

The past two decades have shown that the properties of artificial materials can be tailored to exhibit extraordinary dynamic behavior and properties by cleverly engineering their composition and structure. The development of such metamaterials is a prominent thrust in engineering today.[1–9] The principal focus of metamaterial research is tailored wave control based on the design of subwavelength structure. Elastic waves are of particular interest given the numerous mechanical applications that require their control, such as vibration isolation, impact mitigation, ultrasonography, energy harvesting, and stealth, to name just a few.

Currently, the forefront of research in the control of classical wave motion emerged from a seemingly unrelated field, namely, quantum mechanics, with the development of its non-Hermitian formalism.[10–12] This formalism describes open quantum systems that exchange energy with their environment, resulting with non-orthogonal or even colinear natural modes and degenerate eigenvalues in contrast to Hermitian systems whose natural modes are orthogonal. By drawing analogies between this formalism and those of classical systems,[13–19] researchers have discovered phenomena that defy conventional intuition and have used them to control light,[20–26] sound,[1,27–31] and elastic waves.[32–37] Of all the branches of classical physics, these concepts were least studied in elastodynamics, in spite of (or potentially due to) the fact that elastodynamics exhibits a distinct tensorial richness. In this perspective, we discuss the intricacies and challenges that distinguish non-Hermitian elastodynamics from optics and acoustics, review some of the recent developments in this field, and present an outlook to potential directions from the current state-of-the-art.

II. BACKGROUND

Eigenvalue problems are ubiquitous in physics: they describe the natural modes of the relevant system using a suitable operator. Formally, they may be written as $\mathscr{H}\phi = \alpha\phi$, where $\mathscr{H}$ is an operator acting on the natural mode ϕ, generating its multiplication by the (possibly complex) eigenvalue α. In quantum mechanics, the operator is called the Hamiltonian, which operates on the wave function in the Schrödinger equation. One of the fundamental postulates in quantum mechanics is that the Hamiltonian exhibits a mathematical symmetry termed "Hermiticity," which ensures real eigenvalues and is associated with energy conservation.[38–40] On a basic level, Hermiticity implies that for any two functions ϕ and φ in a function space endowed with an inner product $\langle\cdot,\cdot\rangle$, the Hermitian operator $\mathscr{H}$ satisfies

$$\langle \mathscr{H}\phi, \varphi\rangle = \langle \phi, \mathscr{H}\varphi\rangle. \quad (1)$$

Strikingly, Bender and Boettcher[10] discovered that Hamiltonians with so-called $\mathscr{PT}$ symmetry may also support real spectra. Unlike the eigenmodes of Hermitian operators, the eigenmodes of $\mathscr{PT}$-symmetric operators are no longer orthogonal one to another. In the extreme case, these modes may even coalesce together with their eigenvalues at the so-called "exceptional points" (EPs).[24,41,42] These non-Hermitian degeneracies arise when suitable parameters of non-Hermitian systems are appropriately tuned. Near EPs, the functional dependency of the eigenvalues on these parameters defines self-intersecting Riemann sheets, as shown in Fig. 1.

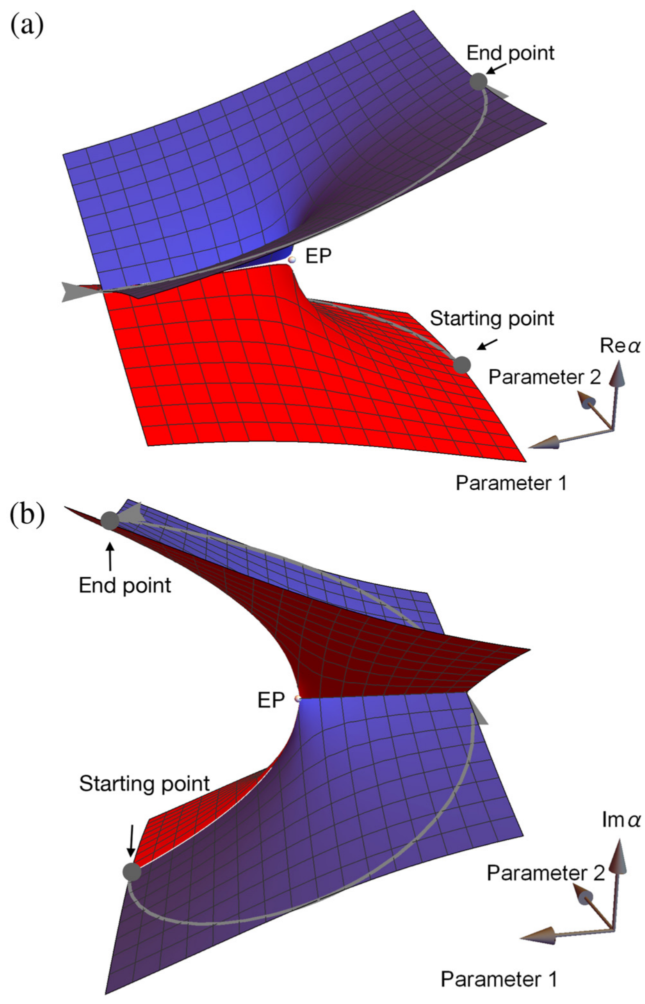

FIG. 1. The (a) real and (b) imaginary parts of the eigenvalues as functions of two suitable parameters of a non-Hermitian operator. For critical values of the two parameters, the Riemann sheets degenerate to a so-called exceptional point. The gray curves depict a loop around the EP, which results in a different state, owing to the multi-valued nature of the Riemann sheet. Conventional research on non-Hermitian systems considers the natural frequencies as the eigenvalues and requires material gain (or loss) to be one of the parameters. In a recent paradigm shift,[43–45] the EPs are designed in the wavevector space, using the unique coupling of elastic wave polarizations. Thus, the EPs are formed without material gain or loss, and the parameters that control the wavevector are the tuning parameters.

EPs and the unique topology around them are the source of fascinating counterintuitive phenomena such as chiral modes,[46] supersensitivity,[47] and unidirectional zero reflection.[48] While the bulk of the research (and the above discussion) considers $\mathscr{PT}$-symmetric Hamiltonians, they belong to a larger class of non-Hermitian Hamiltonians[11,43,49,50] that can exhibit real eigenvalues and EPs at critical values of a suitable parameter set. Understanding that these phenomena rely upon the nature of eigenvalue problems associated with the wave operator has led to their dissemination in other wave physics. We summarize next the fundamental operators in the different classical systems in order to highlight the uniqueness of non-Hermiticity in elastodynamics.

In photonics, Maxwell's equations can be expressed in the form of an eigenproblem[51,52] for the magnetic eigenmodes $\mathbf{H}$ and their frequencies ω^2 (the electric field can be determined subsequently).[53] The corresponding wave operator is a function of the second-order dielectric tensor $\boldsymbol{\epsilon}$ (since the permeability of most materials is very close to the vacuum permeability[52]). Magnetic waves are subjected to the constraint $\nabla \cdot \mathbf{H} = 0$, hence do not support longitudinal modes and are classified as transverse waves.

In acoustics, i.e., in the study of waves in gases and liquids without resistance to shear deformation,[54] the eigenvalue problem can be expressed as a scalar equation for the pressure field, whose operator depends on the mass density ρ, and bulk modulus, K, of the fluid in which waves propagate.[54] The velocity field $\mathbf{v}$ of acoustic waves is subjected to the constraint $\nabla \times \mathbf{v} = \mathbf{0}$, and thus, transverse waves are not supported. Acoustic waves are longitudinal pressure waves, which are also referred to as dilatational or volumetric waves,[55] since they are accompanied by volume change, in contrast to transverse waves, i.e., shear waves that are isochoric.

In solid mechanics, the elastodynamics equations can be formulated as an eigenvalue problem for the time- and space-dependent displacement vector field $\mathbf{u}(\mathbf{x}, t)$ of material points (see the supplementary material). The wave operator is a function of the mass density and the elasticity tensor of the solid, $\mathbf{C}$. Its simplest form is the Christoffel equation[56,57] for bulk plane waves in the $\mathbf{n}$ direction: $k^2\rho^{-1}\boldsymbol{\Gamma}(\mathbf{u}) = \omega^2\mathbf{u}$, where k is the wavenumber and the components of the Christoffel operator are $\Gamma_{ik} = C_{ijkl}n_jn_l$. The first source of richness originates from the dimension of $\mathbf{C}$, which is a fourth order tensor and thus constitutes a larger design space in comparison with photonics and acoustics. Second, unlike the magnetic field (subjected to $\nabla \cdot \mathbf{H} = 0$) and the velocity field in acoustics (subjected to $\nabla \times \mathbf{v} = \mathbf{0}$), the displacement field $\mathbf{u}(\mathbf{x}, t)$ is not subjected to any differential constraint. Thus, even the simplest solids (conservative, isotropic, and homogeneous) support both transverse and longitudinal wave polarizations. These inherent modes can be coupled one with the other, a coupling that gives rise to unique physics.[58]

Most works on non-Hermitian physics focus on problems of the type mentioned above, in which the frequency (squared) is the eigenvalue and corresponds to the propagation of free waves owing to some impulsive excitation, i.e., in the absence of a constant excitation. Assuming real wavenumbers (no radiation losses), the breaking of Hermitian symmetry (i.e., accessing complex frequencies and nonorthogonal eigenmodes) in these problems requires material loss or gain. While optical gain is well established using stimulated emission, generating and controlling elastic gain remains a challenge. We list the recent advances for tackling this challenge in the first part of Sec. III, together with their applications.

Less explored from the perspective of non-Hermitian physics, a dual class of problems involves a formulation where the wavenumber is the eigenvalue and the frequency is a prescribed real quantity (see the supplementary material). This formulation pertains to conservative media with open boundaries and sustained driving sources. Recent progress has utilized the unique coexistence and coupling of wave polarizations in elastodynamics to engineer EPs in such conservative solids, thus circumventing the challenges associated with the design of material gain and loss.[43–45] The second part of Sec. III expands on the works that introduced this approach and how they capitalized on emergent non-Hermitian features to manipulate elastic waves.

III. RECENT DEVELOPMENTS

As discussed in Sec. II, the common approach for breaking Hermitian symmetry requires the careful design of material loss and gain. Piezoelectric materials are therefore natural candidates to achieve non-Hermitian behavior in elastodynamic systems due to their ability to convert electric energy into mechanical energy and vice versa. In a series of works,[59–62] Christensen and collaborators developed a procedure that exploits the acousto-electric effect in piezoelectric semiconductors for material gain, thereby synthesizing $\mathcal{PT}$-symmetric elastic media[32] and breaking elastodynamic reciprocity.[34] When an acoustic field impinges on a piezoelectric semiconductor slab, a coherently oscillating electric charge is created. Superposing a sufficiently high DC electric field corresponding to a supersonic carrier drift speed leads to sound amplification by virtue of phonon emission, an effect known as acoustic Cherenkov radiation. In particular, this acousto-electric effect has been employed for two demonstrations. First, stacking multiple piezoelectric semiconductors was shown to generate a system with balanced loss and gain with associated non-Hermitian properties, if the layers are adequately electrically loaded.[32] Next, a theoretical non-Hermitian Su–Schrieffer–Heeger demonstration was made using the same approach. In that case, the non-Hermitian skin effect and the ensuing failure of both the Bloch band topology were demonstrated.[62] We note, however, that experimental demonstration of the latter theoretical prediction has not yet been provided.

A more conventional approach to use piezoelectric materials is to shunt them through electrical circuits of resistors, capacitors, and inductors,[63] which collectively generate tunable elastic gain loss.[33,63] Assouar's group proposed a tunable $\mathcal{PT}$-symmetric elastic beam based on such shunted piezoelectric elements.[33] They later employed this concept for negative refraction of flexural waves.[35] Similarly, Huang's group employed shunted piezoelectric patches attached to a $\mathcal{PT}$-symmetric elastic beam that displayed asymmetric flexural wave scattering.[64]

Ruzzene and colleagues have also used shunted piezoelectric arrays to generate spatiotemporal modulation of the elastic modulus, thereby breaking reciprocal elastic wave transport for flexural waves.[65,66] This approach was later employed by Erturk and colleagues.[67] to demonstrate the formation of a resonant EPs. Such materials that violate time invariance can exhibit a wide array of nonreciprocal wave phenomena such as one-way wave amplification and attenuation.[68–70] From a modeling perspective, the constitutive parameters and phase velocity become time-dependent. These works specifically considered modulations in the form of progressive periodic waves ("pump waves"), which create a spatiotemporal bias and nonreciprocal wave motion as a function of the modulation. Small-amplitude medium-speed modulations lead to a Bragg scattering regime; faster modulations can be described by an equivalent medium with effective properties; and slower but higher-amplitude modulations lead to an adiabatic regime.[70] This nonreciprocal wave propagation is fundamentally different from one-way Bragg reflection, but can still be leveraged for the purposes of one-way mirroring, especially at low frequencies where wide bandgaps are seldom available. In this scenario, Hooke's law no longer applies and must be replaced by constitutive relations of the Willis form,[71–73] which state that the stress and linear momentum depend both the strain *and* velocity fields (more precisely, the stress depends on the acceleration or time-history of the velocity, such that there is no contribution to the stress if the velocity is constant in time[74]). The Willis parameters in the case of spatiotemporal modulations capture the nonreciprocal nature of the modulated microstructures.[68] Indeed, Willis materials offer another platform for breaking the Hermitian nature,[75] as we discuss later.

Collectively, the works above have tackled the important challenge of generating gain in elastic materials in order to break temporal Hermiticity. However, gain is not a necessity for realizing EPs and some of their associated features are accessible using loss alone, an inherent feature of viscoelastic materials. Indeed, Shmuel and Moiseyev[76] showed that the introduction of a viscoelastic part in an elastic slab may generate EP of two modes (EP2) in the spectrum of the assembly. They applied non-Hermitian perturbation theory previously developed in quantum mechanics to determine the conditions under which this occurs and employed the resultant topology around the EP for mass sensing with enhanced sensitivity. Similarly, Domínguez-Rocha *et al.*[77] used differential loss instead of gain and loss to form EP2. While the modes they analyzed were structural modes (torsional modes of pillars) rather than elastodynamics modes, their work is one of the few experimental demonstrations of the sensitive frequency splitting near EP of a mechanical system.

The nonconservative nature of the media presented so far manifests itself through time-dependent constitutive equations and hence complex elastic moduli. These, in turn, yield complex frequencies for the Christoffel equation. Another type of nonconservative media are Cauchy-elastic media:[78] the work done by the stress in such media generally depends on the deformation path, and therefore cannot be derived from a strain energy function. As a result, the elasticity tensor no longer exhibits the major symmetry that is needed to ensure real frequencies in the Christoffel equation. Under the name "odd elasticity," Vitelli and colleagues[79,80] set forth a model of active material with nonconservative microscopic interactions that generate odd elastic moduli and studied the non-Hermitian elastodynamics of such materials with odd elasticity (top circle in Fig. 2). They demonstrated that in isotropic solids with odd elasticity, the wave polarizations (eigenmodes) are no longer orthogonal and may even become colinear for a threshold value of the elastic moduli. This EP marks the transition to odd-elastic waves with circular polarization.[79]

Active material systems displaying odd elasticity have been experimentally demonstrated by introducing piezoelectric elements and motors controlled by electrical circuits, or spinning networks into host media.[81,82] Chen *et al.*[82] considered 1D non-Hermitian metamaterial in which each unit cell consists of three piezoelectric patches mounted on a steel beam. The metabeam was assumed to have two modes of deformation including bending and shearing. These modes of deformation in turn induce a shear stress σ and a bending moment, M. The crucial difference between this non-Hermitian beam and a traditional

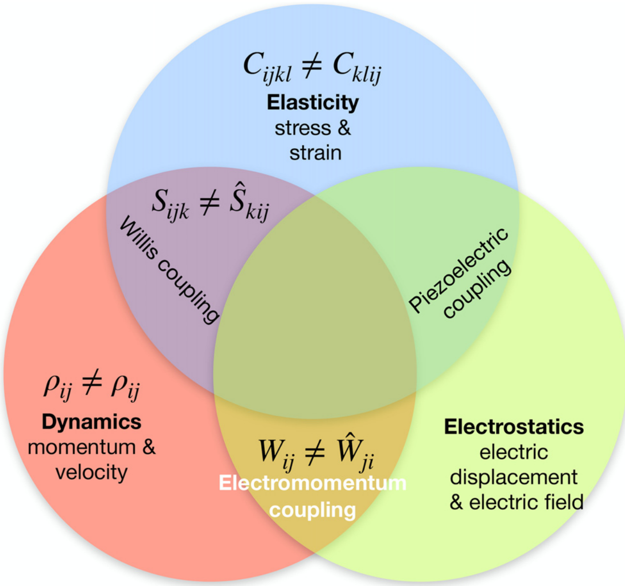

FIG. 2. Possible non-Hermitian constitutive operators in elastodynamics. Elastic materials may exhibit odd elastic tensor ($\mathbf{C} = -\mathbf{C}^{\mathrm{T}}$), Willis materials may exhibit odd Willis- ($\hat{\mathbf{S}} = -\mathbf{S}^{\mathrm{T}}$) and density ($\boldsymbol{\rho} = -\boldsymbol{\rho}^{\mathrm{T}}$) tensors, and electromomentum materials may exhibit odd electromomentum tensors ($\hat{\mathbf{W}} = -\mathbf{W}^{\mathrm{T}}$).

beam is the presence of internal energy sources that violate energy conservation. By designing the feedback to create non-reciprocal coupling between the elongation s and shear b, the constitutive relation of the metabeam takes the form of odd elasticity,

$$\begin{bmatrix} \sigma \\ M \end{bmatrix} = \begin{bmatrix} \mu & P \\ 0 & B \end{bmatrix} \begin{bmatrix} s \\ b \end{bmatrix}, \tag{2}$$

i.e., the electronic feedback between the piezoelectric patches induces a new modulus P, in addition to the shear and bending moduli μ and B, respectively. Because the energy differential is $\delta\psi = \sigma\delta s + M\delta b$, the asymmetric part of the matrix corresponds to a violation of Maxwell–Betti reciprocity. The parity-violating, nonconservative modulus P also induces unidirectional wave amplification. For finite structures, these non-Hermitian systems with odd elasticity exhibit non-Hermitian 1D and 2D skin effect and non-Hermitian Rayleigh wave propagation.[82,83] The skin effect in the non-Hermitian system is defined as the case where bulk modes behave as the skin modes collapsing at the open boundaries. Recently, the design of active microstructure for a 2D non-Hermitian odd plate with odd density through feed-forward interactions was suggested to explore a series of unconventional wave phenomena when conventional Timoshenko plate mechanics meets with non-Hermiticity.[84,85] Those works numerically and experimentally demonstrated nonreciprocal wave amplification and attenuation phenomena along with direction-dependent dispersion control of flexural waves in 2D odd plates.

The approaches mentioned so far to design non-Hermitian elastodynamics can be generalized using Willis materials,[71,72,86] which exhibit a different conversion of strain energy and kinetic energy relative to conventional materials.[87] This distinct mechanism is reflected by the Willis tensors $\mathbf{S}$ and $\hat{\mathbf{S}}$ and tensorial mass density $\boldsymbol{\rho}$ appearing in nonlocal constitutive equations, analogous to the bianisotropic equations in electromagnetism.[87,88] In the spatially local limit, time-harmonic case, the constitutive (Milton-Briane-Willis[88,89]) equations take the form $\sigma_{ij} = C_{ijkl}u_{k,l} - S_{ijk}\ddot{u}_k/(i\omega)$ and $p_i = \rho_{ij}\dot{u}_j - \hat{S}_{ijk}\dot{u}_{j,k}/(i\omega)$. Extending the concept of odd elasticity to odd mass density and odd Willis tensors thus breaks Hermitian symmetry in the Christoffel equation, giving rise to complex frequencies. This translates to designing a Willis material such that one of the symmetries $\rho_{ij} = \rho_{ji}$, $\hat{S}_{ijk} = S_{kij}$ is broken (left circle in Fig. 2). Huang and collaborators[85] designed and experimentally realized such active microstructure that gives rise to odd mass density. They further demonstrated the formation of EPs of transition between stable and unstable waves, directional wave amplification, and non-Hermitian skin effect in the medium. In another collaboration,[90] they experimentally realized odd Willis couplings in active flexural media using piezoelectric sensor–actuator pairs controlled with digital circuits, giving rise to nonreciprocal wave propagation. Li and collaborators[91] introduced loss to a Willis beam and showed the formation of EP in the scattering matrix for flexural waves, at which there occurs unidirectional zero reflection.

Building upon these works probing gain and loss in elastic materials, Psiachos and Sigalas[36] showed that coupled transverse-longitudinal elastic waves scatter asymmetrically from alternating layers with gain and loss. Importantly, they later realized that the coupling between transverse and longitudinal waves is sufficient to generate asymmetric scattering even without gain and loss.[92]

All the developments that are listed above rely upon material gain, loss or their combination in order to access non-Hermitian features in elastodynamics. Recently, Shmuel's group[43] introduced a distinct paradigm to eliminate the need for material gain or loss by breaking *spatial* Hermitian symmetry using the tensorial nature that is unique to elastodynamics. From a mathematical perspective, they designed the non-Hermitian part of the operator for the wavenumbers instead of the operator for the natural frequencies (see the supplementary material). From a mechanical perspective, the analysis is of a conservative solid with open boundaries and sustained driving source. More specifically, they revisited the canonical scattering problem of monochromatic plane waves impinging on a semi-infinite periodic laminate made of two conservative isotropic materials.[93] By designing the unit cell, they tuned the energy transfer between the two elastic polarizations at the material interfaces such that two of the forward (quasi-transverse and quasi-longitudinal) Bloch modes coalesce inside the Brillouin zone[43] [Fig. 3(a) (Multimedia view)]. This is in sharp contrast to the well-known scenario of degeneracies at the band edges between forward and backward Bloch modes of the same polarization.[94]

The EPs that are associated with this non-Hermitian eigenmode degeneracy are formed in the complex wavevector space, in contrast to the gain and complex frequency parameter space as in the works discussed above. These non-resonant EPs are surrounded by a self-intersecting Riemann surface [Fig. 3(b)], whose curvature is related to the energy flow and therefore the unique topology of these surfaces can be leveraged for anomalous energy transport. For example, Lustig *et al.*[43] showed that these spatial EPs give rise to negative refraction [Fig. 3(c)], as they are the crossing of branches of opposite slopes. This phenomenon was experimentally demonstrated recently by Li *et al.*[95] The key point is that, contrary to previous works discussed above, the parameters of the Riemann surface about the non-resonant EPs do not correspond to material gain or loss, but, e.g., the angle of the incident wave as depicted in the left half-space in Fig. 3(c). Subsequently, Srivastava's group[44] developed designated tools to further analyze the non-Hermitian operator, its EPs and scattering spectrum, suggesting its connection to resonance trapping.

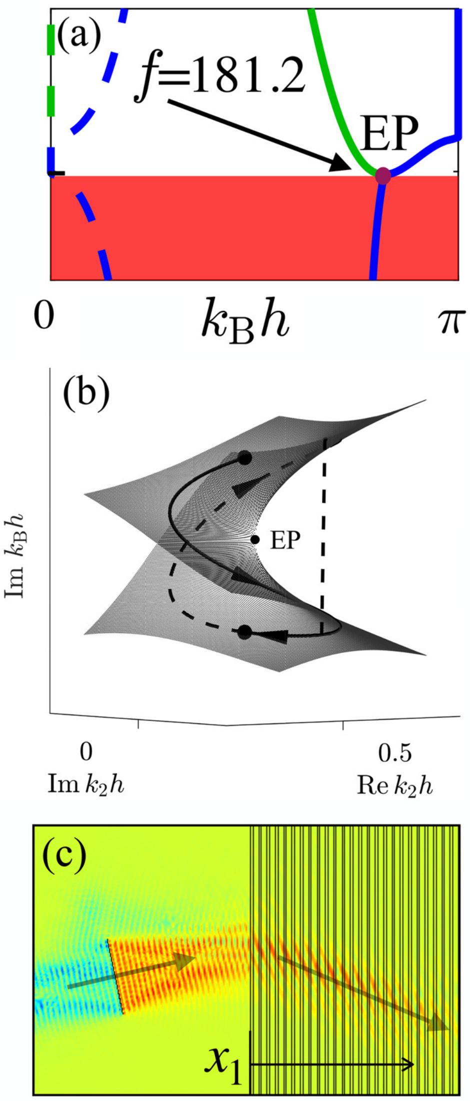

FIG. 3. (a) The coalescence of two Bloch modes of the exemplary conservative laminate considered by Lustig *et al.*[43] The EPs are shown in a cut in the frequency-Bloch wavenumber (f, k_{B}) diagram for normalized vertical wavenumber $k_2h = 0.5$ about 181.2 kHz. Re k_{B} and Im k_{B} are shown in solid and dashed curves, respectively. The real and imaginary parts of a certain branch are plotted using the same color. (b) One of the Riemann sheets associated with this EP in the wavevector space. The solid and dashed curves illustrate the asymmetric nature of a loop around the EP. One direction (solid curve) is adiabatic and the other (dashed curve) is non-adiabatic, resulting in a different state at the end of the loop. (c) Full-wave finite element simulations of the axial energy flux in the pertinent scattering problem near the EP. The energy flux is preserved in the laminate and exhibits negative refraction. Figure adapted from Lustig *et al.*[43] Multimedia available online.

In a subsequent work,[45] Shmuel's group designed the coalescence of three Bloch modes (EP3) by replacing one of the isotropic materials in the unit cell with anisotropic one. This non-resonant higher degeneracy further expands the toolbox for elastic wave shaping by forming waves with zero axial group velocity and finite transmittance (known as "axially frozen modes"[96]), which can even reach unity [Fig. 4(b) (Multimedia view)]. Notably, these modes, which were previously discovered in 3D dielectric laminates,[96] are now accessible in simpler, planar settings in elastodynamics, thanks to its distinct tensorial richness. The theoretical model used is an idealization of any experimental realization, and therefore, its results are limited by practical deviations such as material defects and finiteness of the specimen. The results are also limited by nonlinear effects brought by frozen modes with inherent large amplitude.

Additional useful phenomena are accessible by combining non-resonant EPs with gain and loss. For example, such combination enables the encirclement of non-resonant EPs in a suitable parameter space, a process that is known to be asymmetrically non-adiabatic.[97–99] This means that the physical system will follow its instantaneous eigenmode when subjected to a loop in the parameter space, only along one of the two possible orientations of the loop. The solid (counterclockwise) and dashed (clockwise) curves in Fig. 3(b) demonstrate such adiabatic and non-adiabatic state evolution, respectively. Elbaz *et al.*[100] carried out such an encirclement around non-resonant EP2 of forward and backward longitudinal Bloch waves in elastic laminates, using spatial modulation of gain and loss. They discovered that the starting point of the loop governs several unusual features. For example, the laminate may act as a source or a sink of energy, exhibit reflectance greater than

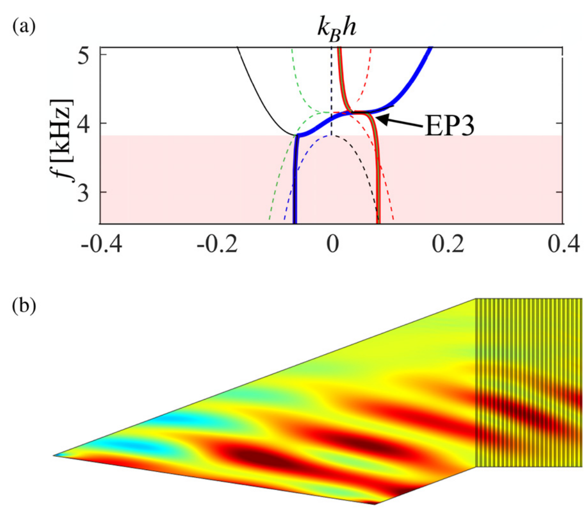

FIG. 4. (a) Coalescence of three Bloch modes of the exemplary conservative laminate considered by Fishman *et al.*[45] This EP3 is shown in a cut in the (f, k_{B}) diagram for normalized vertical wavenumber $k_2h = 0.15$ about 4.15 kHz. Re k_{B} and Im k_{B} are shown in solid and dashed curves, respectively. The real and imaginary parts of a certain branch are plotted using the same color. (b) Full-wave finite element simulations of the axial energy flux in the pertinent scattering problem near the EP, giving rise to axially frozen modes with unity transmittance. Figure adapted from Fishman *et al.*[45] Multimedia available online.

unity, and accommodate spatial asymmetry in the energy flow with respect to the incidence direction, depending on that starting point. Encircling EPs provides another unexpected benefit in the form of a highly efficient algorithm for sorting eigenvalue bands, as observed by Lu and Srivastava,[101] who exploited EPs to distinguish real crossings from level repulsion zones in the real phononic spectrum.

IV. FUTURE DIRECTIONS

A future avenue that has been thoroughly explored in acoustics and photonics concerns non-Hermitian topology.[102] Non-Hermitian topology refers to the study of topological properties in systems that do not satisfy the Hermitian symmetry. In other words, non-Hermitian topology explores the emergence of topological phenomena in systems containing lossy and amplifying components. One of the key concepts in this field is the non-Hermitian skin effect. This effect refers to the near-complete localization of all eigenstates at the boundaries of a system owing to complex winding structures in non-Hermitian system's eigenvalues. This phenomenon is driven by nonreciprocal hopping, where particles preferentially move in specific directions, breaking translational symmetry. As a result, the conventional Bloch band theory fails to describe the system, leading to an unusual localization of bulk states at the boundaries. The effect has intriguing implications in the study of open quantum systems and has broadened our understanding of localization phenomena, especially in a topological contexts, even considering mechanical vibrations. Using planar mechanical metamaterials, several numerical[103] and experimental[84,104] works demonstrated how the eigenstates become localized at the boundary rather than uniformly distributed throughout the bulk. Targeting elastodynamic non-Hermitian bulk states at interfaces beyond the plane remains an intriguing area of research to pursue. Other exciting directions include combining lattice symmetry and non-Hermiticity and other phenomena that are elusive in elastodynamics such as non-Hermitian Weyl exceptional rings, non-Hermitian higher-order topology with vibrating complex corner states, and non-Abelian permutations[102] of mechanical states.

A completely uncharted direction is the breaking of Hermitian symmetry in materials that display electromomentum couplings.[40,105–109] The electromomentum couplings were theoretically discovered by Pernas-Salomón and Shmuel[105] using a generalization of Willis' dynamic homogenization method[72] to composites made of constituents that mechanically respond to non-mechanical stimuli. For the case of piezoelectric constituents, the method revealed that the electric displacement field ($\mathbf{D}$) constitutively depends on velocity and that the macroscopic linear momentum constitutively depends on the electric field ($-\nabla\phi$). These two couplings are captured in terms of two second-order tensors ($\mathbf{W}$ and $\hat{\mathbf{W}}$) called the electromomentum tensors, in direct analogy with the Willis tensors. These coupling tensors capture macroscale effects resulting from spatial symmetry-breaking on the microscale and nonlocal interactions.[40,105,106] For the Willis couplings, it is the asymmetry in the elastic impedance that forms their local part, while for the electromomentum coupling, it is the asymmetry in the piezoelectric profile. Notably, the piezoelectric couplings also emerge from broken spatial symmetry in the atomic structure. Depending on the circuit conditions, the electromomentum coupling can modify the phase velocity (like the piezoelectric coupling) and introduce a directional phase angle to the characteristic elastic impedance (like the Willis coupling). In symbolic matrix notation, the spatially local limit of these constitutive equations is

$$\begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{C} & \hat{\mathbf{B}} & \hat{\mathbf{S}} \\ \mathbf{B} & -\mathbf{A} & \mathbf{W} \\ \mathbf{S} & \hat{\mathbf{W}} & \boldsymbol{\rho} \end{pmatrix} \begin{pmatrix} \nabla\mathbf{u} \\ \nabla\phi \\ \dot{\mathbf{u}} \end{pmatrix}, \quad (3)$$

where $\mathbf{A}$ is the permittivity tensor and $\mathbf{B}$ and $\hat{\mathbf{B}}$ capture the direct- and inverse piezoelectric effect, respectively; collectively, these equations take a trianisotropic form.[110] Note that the constituents of such composites are piezoelectric and do not display Willis nor electromomentum coupling. To date, studies[111–114] have been restricted to materials with electromomentum tensors that are Hermitian adjoints,[40] which implies that the local electromomentum tensors satisfy $\hat{W}_{ij} = W_{ji}$. However, considering the works listed in Sec. III that exploit piezoelectricity to generate material gain, this platform naturally lends itself to non-Hermitian physics, the simplest of which is of odd electromomentum tensors ($W_{ij} = -\hat{W}_{ji}$, right circle in Fig. 2). The implications of breaking this symmetry on elastic waves and how to realize designs with odd electromomentum tensors are yet to be explored and constitute excellent candidates to extend this approach for elastic wave control.

SUPPLEMENTARY MATERIAL

See the supplementary material for the mathematical framework for non-Hermitian operators in elastodynamics, with both frequencies and wavenumbers considered as eigenvalues.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Johan Christensen: Conceptualization (supporting); Writing – original draft (supporting); Writing – review & editing (supporting). **Michael R. Haberman:** Conceptualization (supporting); Writing – original draft (supporting); Writing – review & editing (supporting). **Ankit Srivastava:** Formal analysis (supporting); Funding acquisition (equal); Writing – original draft (supporting); Writing – review & editing (supporting). **Guoliang Huang:** Funding acquisition (equal); Writing – original draft (supporting); Writing – review & editing (supporting). **Gal Shmuel:** Conceptualization (lead); Formal analysis (lead); Funding acquisition (equal); Investigation (lead); Project administration (lead); Writing – original draft (lead); Writing – review & editing (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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